CSI 30 SPRING 24 PROF. PINEIRO FINAL EXAM PREPARATION

1. Write the propositions using simpler propositions and logical connectives and determine, if possible, true or false.
   (a) If $4 \geq 1$ then $x > 10$.
   (b) If $x < 5$ then $2 \leq 25$
   (c) If $3 \geq -2$ then $4 > 7$.
   (d) If $7 < 1$ then $14 > 10$

   Ans: (a) Since $p : 4 \geq 1$ is True, the proposition $p \to q$ can be **True or False** depending on the truth value of $q : x > 10$. **We cannot determine**.
   (b) Since $q : 2 \leq 25$ is true, the proposition $p \to q$ will be **True** regardless of the truth value of $p : x < 5$
   (c) Now, $p : 3 \geq -2$ is True and $q : 4 > 7$ is False, therefore $p \to q$ is **False**.
   (d) Since $p : 7 < 1$ is False and $14 > 10$ is true, the proposition $p \to q$ is **True**. If we were to change $p : 7 < 1$ by $x < 1$ and $q : 14 > 10$ by $10 < 14$, then it will depend on the value of x.

2. Show that the compound proposition $(p \wedge (p \to q)) \to q$ is a tautology **using a truth table**.

p	q	$p \to q$	$(p \wedge (p \to q))$	$(p \wedge (p \to q)) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

3. Consider the proposition 'Some people do go good in every situation'.
   (a) Write the proposition using quantifiers. Express clearly the domain of your variables.
   (b) Negate the proposition using quantifiers and express your answer in such a way that no quantifier is immediately preceded by a negation.

   Ans: Consider x with domain all people and y in the domain of all possible situations. We take that Good(x, y) means 'x do good on situation y'. With this, we can write the proposition as:

$$\exists x \forall y \text{Good}(x, y).$$

   The negation will be: 'Nobody does good in every situation' or 'For every person, there is a no good situation'. With quantifiers:

$$\forall x \exists y \neg \text{Good}(x, y).$$

4. Let S be the set $S = \{\{1\}, \{2\}, 3\}$. Determine True or False for the following statements:
   (a) $\{2\} \in S$. **True**, since $\{2\}$ is an element of A. The set A has three elements $\{1\}, \{2\}$ and 3.
   (b) $\{2\} \subset S$. **False**, since $\{2\}$ is an element and not a subset of A.
   (c) $\{3\} \subset S$. **True**, since 3 is an element of A and therefore $\{3\}$ is subset of A with one element.

5. Consider the sets $A = \{0, 1, 2\}$ and $B = \{a, b, c\}$.
   (a) List the elements in $\mathcal{P}(B)$.

$$\mathcal{P}(B) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}.$$

   (b) Build $A \times B$.

$$A \times B = \{(0, a), (0, b), (0, c), (1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

   (c) Give an example of a function from A to B. Is your function one-to-one? onto?

$$f = \{(0, a), (1, a), (2, b)\}.$$

   Not one-to-one. Not onto.
   (d) Give an example of a **relation from** A to B that **is not a function**.

$$r = \{(0, a), (1, a), (2, b)(2, a)\}.$$

6. (5 points) Suppose that the universe $\mathbb{U} = \{\text{prime numbers } \leq 30\}$, $A = \{7, 13, 29\}$ and $B = \{5, 11, 13, 17, 29\}$.
   (a) Determine $A \cup B$ and $|A \cup B|$.

$$A \cup B = \{7, 13, 29, 5, 11, 17\}.$$

   $|A \cup B| = 6$.
   (b) Determine $A \cap B$ and $|A \cap B|$.

$$A \cap B = \{13, 29\}$$

   $|A \cap B| = 2$.
   (c) Determine $\bar{A}$ and $|\bar{A}|$.

$$\bar{A} = \{2, 3, 5, 11, 17, 19, 23\}$$

   $|\bar{A}| = 7$.
   (d) Determine $A - B$.

$$A - B = \{7\}$$

(e) Represent A with a bit string of length 10 using in $\mathbb{U}$ the increasing order.
Ans: The universe $\mathbb{U}$ is $\mathbb{U} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$ and therefore:

$$A = (0, 0, 0, 1, 0, 1, 0, 0, 0, 1).$$

7. (5 points) Given the algorithm:

procedure partial($a_1, a_2, a_3, \ldots, a_n$: integers)
$sum_1 := 0$
$sum_2 := 0$
for i := 1 to n
  if ($a_i > 0$): $sum_1 := sum_1 + a_i$
  if ($a_i < 0$): $sum_2 := sum2 + a_i$
return(sum_1, sum_2)

For the set of values $\{-4, 5, -7, 2, 9, 0, -2\}$ as input for the above algorithm, what values of $prod_1$ and $prod_2$ that will be returned?
Ans: $sum_1 = 5 + 2 + 9 = 16$ and $sum_2 = -4 - 7 - 2 = -13$.

8. Find the greatest common divisor GCD(578, 153) using the Euclidean Algorithm:

procedure GCD (a, b: positive integers):
x := a
y := b
While $y \neq 0$:
  r := x mod y
  x := y
  y := r
return('The GCD is' : x)
Find integers t, s such that $578t + 153s =$ GCD(578, 153).

n	0	1	2	3	4	5
q			3	1	3	2
r	578	153	119	34	17	0
s	1	0	1	-1	4	
t	0	1	-3	4	-15	

We have $17 = 4(578) - 15(153)$, or $t = 4$ and $s = -15$.

9. Consider the numbers $n = 347$ and $m = (1D3C)_{16}$. (Use $A = 10, B = 11, \ldots, F = 15$).
   (a) Find the representation of n is base 16. Ans: $n = (15B)_{16}$.
   (b) Compute the decimal representation of m. Ans: 7484.
   (c) Find $m + n$ in base 16. Ans: $1E97$.
   (d) Find $m + n$ in base 10. Ans: 7831

10. In how many ways can the letters of the word 'AYAYIYO' be arranged?
Ans: There 7 letters in total, but with 3 identical Y and 2 identical A, therefore:

$$\frac{7!}{2!3!} = 420.$$

11. What is the coefficient of x^2y^4 in the expansion of the binomial $(3x - 5y)^6$.
Ans: The coefficient will be:

$$\binom{6}{2} 3^2(-5)^4 = 15(9)(625) = 84,375.$$

12. How many bit-strings of length 6 contain not consecutive zeroes?
Ans: Let a_n denote the number of bit-strings of length n without consecutive zeroes. We can compute directly $a_0 = 1$ and $a_1 = 2$. The relation is $a_{n+1} = a_n + a_{n-1}$, since any bit sequence of length $n+1$ is obtained by either adding a 1 at the end of a sequence of length n without consecutive zeroes or adding 10 at the end of a good sequence with length $n-1$. Since any sequence must end in a 0 or a 1, we count in this way all sequences without consecutive zeroes. The values for small n can be easily computed: $a_0 = 1$, $a_1 = 2$, $a_2 = a_0 + a_1 = 3$, $a_3 = 5$, $a_4 = 8$, $a_5 = 13$ and $a_6 = 21$.

13. How many integers from 1 to 100 are divisible by 2 or by 3?
Ans: In general the formula for how many numbers are divisible by k in the range $[0, n]$ is $\lfloor n/k \rfloor + 1$. So, by the general addition rule:

$$D(2 \text{ or } 3) = D(2) + D(3) - D(2 \text{ and } 3)$$

$$D(2 \text{ or } 3) = D(2) + D(3) - D(6) = 51 + 34 - 17 = 68.$$

We have used the fact that being divisible by 2 and 3 is equivalent to be divisible by 6.

14. Consider the following generator of pseudo-random numbers:

$$x_n = (5x_{n-1} + 7) \bmod 12, \quad \text{with seed} \quad x_0 = 4.$$

What sequence of pseudo-random numbers does it generate?
Ans: The sequence generate is $\{4, 3, 10, 9\}$.

15. What is the probability that a 5-card poker hand contains exactly three hearts?
Ans: Since a deck of card has 52 cards in total, the total amount of five cards hands is $\binom{52}{5}$. Out of those, the hands that contain exactly three hearts are given by the product $\binom{13}{3}\binom{39}{2}$ obtained when we take 3 from the 13 hearts and 2 from the remaining 39 cards. The probability is:

$$P = \frac{\binom{13}{3}\binom{39}{2}}{\binom{52}{5}} \approx .0815.$$

16. Calculate $\mathrm{P}(2000, 3)$ and $\binom{2000}{3}$.

Ans: The permutations are $\mathrm{P}(2000, 3) = \dfrac{2000!}{1997!} = (2000)(1999)(1998) = 7,988,004,000$ and the combinations $\binom{2000}{3} = \dfrac{\mathrm{P}(2000, 3)}{3!} = 1,331,334,000$.

Suppose that S is a set of 2000 elements.

The number $\mathrm{P}(2000, 3)$ represents the number of triples (x, y, z) using elements of S without repetitions.

At the same time, the number $\binom{2000}{3}$ represents the number of subsets of S having exactly 3 elements.